4.1: Vector Spaces and subspaces

Key idea: the algebraic properties we have observed about vectors in $\mathbb{R}^n$ hold of many other mathematical objects. We define any such setting as a vector space and use what we have learned about $\mathbb{R}^n$ to study, for example, all continuous functions.

Notice a good deal of the work we have done thus far has relied on the algebraic properties of $\mathbb{R}^n$, we can generalize this work to many other mathematical systems which satisfy the same properties. We call such a system a vector space and ensure it satisfies the following properties.

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

Notice how intuitive these properties are in $\mathbb{R}^n$: 1-5 addition acts "as it should" ($\vec{0}$ is identity, $-\vec{u}$ is inverse, commutativity, associativity)

6-10 scalar multiplication acts "as it should" (distribution, associativity, identity)

The power of linear algebra comes from recognizing these properties elsewhere and then using all of the theory we've developed in the previous chapters to study this new setting (e.g. continuous functions)

Before we can do this however, we need practice recognizing these properties in several examples.

We work through "examples of vector spaces and subspaces."